ON A VARIANT OF PILLAI'S PROBLEM

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Abstract. In this paper, we find all integers c having at least two representations as a difference between a Fibonacci number and a Tribonacci number.

1. Introduction

Pillai's famous conjecture first formulated in [11] states that the Diophantine equation

$$a^x - b^y = c \tag{1}$$

has for any fixed integer $c > 0$ at most finitely many solutions a, b, x, y in positive integers. This conjecture is still open for all $c \neq 1$. Note that the case $c = 1$ is Catalan's conjecture which has been solved by Mihăilescu [10]. If we leave a, b and c fixed, then much more is known about the solutions (x, y). For instance Pillai [11] showed that if c is larger than some constant depending on a and b, then Diophantine equation (1) has at most one solution. In particular, he conjectured that in the case that $a = 3$ and $b = 2$ Diophantine equation (1) has at most one solution if $c > 13$. This conjecture was confirmed by Stroeker and Tijdeman [12] and their result was further improved by Bennett [5], who showed that for fixed a, b and c equation (1) has at most two solutions.

Recently Ddamulira, Luca and Rakotomalala [7] considered the Diophantine equation

$$F_n - 2^m = c, \tag{2}$$

where c is a fixed integer and $\{F_n\}_{n\geqslant 0}$ is the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 0$. This type of equation can be seen as a variation of Pillai's equation. However Ddamulira et.al. proved that the only integers c having at least two representations of the form $F_n - 2^m$ are contained in the set $\mathcal{C} = \{0, 1, -3, 5, -11, -30, 85\}$. Moreover, they computed for all $c \in \mathcal{C}$ all representations of the from (2).

The purpose of this paper is to consider a related problem. Denote by $\{T_m\}_{m\geqslant 0}$ the sequence of Tribonacci numbers given by $T_0 = 0$, $T_1 = T_2 = 1$ and $T_{m+3} = T_{m+2} + T_{m+1} + T_m$ for all $m \geqslant 0$. The main result of our paper is to find all nonzero integers c admitting at least two representations of the form $F_n - T_m$ for some positive integers n and m. It is assumed that representations with $n \in \{1, 2\}$ (for which $F_1 = F_2 = 1$) as well as representations with $m \in \{1, 2\}$ (for which

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$T_1 = T_2$) count as one representation to avoid trivial parametric families such as $1-1 = F_1 - T_1 = F_2 - T_1 = F_1 - T_2 = F_2 - T_2$. Therefore we assume that $n \geqslant 2$ and $m \geqslant 2$. We prove the following theorem:

Theorem 1. *The only integers c having at least two representations of the form $F_n - T_m$ come from the set*

$$\mathcal{C} = \{0, 1, -1, -2, -3, 4, -5, 6, 8, -10, 11, -11, -22, -23, -41, -60, -271\}.$$

Furthermore, for each $c \in \mathcal{C}$ all representations of the form $c = F_n - T_m$ with integers $n \geqslant 2$ and $m \geqslant 2$ are:

$$\begin{aligned}
0 &= 1-1 = 2-2 = 13-13 \ (= F_2 - T_2 = F_3 - T_3 = F_7 - T_6),\\
1 &= 2-1 = 3-2 = 5-4 = 8-7\\
&\qquad\qquad (= F_3 - T_2 = F_4 - T_3 = F_5 - T_4 = F_6 - T_5),\\
-1 &= 1-2 = 3-4 \ (= F_2 - T_3 = F_4 - T_4),\\
-2 &= 2-4 = 5-7 \ (= F_3 - T_4 = F_5 - T_5),\\
-3 &= 1-4 = 21-24 \ (= F_2 - T_4 = F_8 - T_7),\\
4 &= 5-1 = 8-4 \ (= F_5 - T_2 = F_6 - T_4),\\
-5 &= 2-7 = 8-13 = 144-149 \ (= F_3 - T_5 = F_6 - T_6 = F_{12} - T_{10}),\\
6 &= 8-2 = 13-7 \ (= F_6 - T_3 = F_7 - T_5),\\
8 &= 21-13 = 89-81 \ (= F_8 - T_6 = F_{11} - T_9),\\
-10 &= 3-13 = 34-44 \ (= F_4 - T_6 = F_9 - T_8),\\
11 &= 13-2 = 55-44 \ (= F_7 - T_3 = F_{10} - T_8),\\
-11 &= 2-13 = 13-24 \ (= F_3 - T_6 = F_7 - T_7),\\
-22 &= 2-24 = 121393-121415 \ (= F_3 - T_7 = F_{26} - T_{21}),\\
-23 &= 1-24 = 21-44 \ (= F_2 - T_7 = F_8 - T_8),\\
-41 &= 3-44 = 233-274 \ (= F_4 - T_8 = F_{13} - T_{11}),\\
-60 &= 21-81 = 89-149 \ (= F_8 - T_9 = F_{11} - T_{10}),\\
-271 &= 3-274 = 233-504 \ (= F_4 - T_{11} = F_{13} - T_{12}).
\end{aligned}$$

2. Preliminaries

In this section, the result of linear forms in logarithms by Baker and Wüstholz [3] is stated. Besides, we state a lemma used by Ddamulira et.al. [7], which is a slight variation of a result due to Dujella and Pethő [8], of which is a generalization of a result due to Baker and Davenport [2]. Both results will be used in the proof of Theorem 1.

2.1. **A lower bound for linear forms in logarithms of algebraic numbers.**

In 1993, Baker and Wüstholz [3] obtained an explicit bound for linear forms in logarithms with a linear dependence on $\log B$. It is a vast improvement compared with lower bounds with a dependence on higher powers of $\log B$ in preceding publications by other mathematicians in particular Baker's original results [1]. The final structure for the lower bound for linear forms in logarithms without an explicit determination of the constant $C(k, d)$ involved has been established by Wüstholz [13] and the precise determination of that constant is the central aspect of [3] (see also [4]). The improvement was mainly due to the use of the analytic subgroup theorem established by Wüstholz [14]. We shall now state the result of Baker and Wüstholz.

Denote by $\alpha_1, \ldots, \alpha_k$ algebraic numbers, not 0 or 1, and by $\log \alpha_1, \ldots, \log \alpha_k$ a fixed determination of their logarithms. Let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$ and let $d = [K : \mathbb{Q}]$ be the degree of K over $\mathbb{Q}$. For any $\alpha \in K$, suppose that its minimal polynomial over the integers is

$$g(x) = a_0 x^\delta + a_1 x^{\delta-1} + \ldots + a_\delta = a_0 \prod_{j=1}^{\delta} (x - \alpha^{(j)}).$$

The absolute logarithmic Weil height of α is defined as

$$h_0(\alpha) = \frac{1}{\delta}\left(\log |a_0| + \sum_{j=1}^{\delta} \log\left(\max\{|\alpha^{(j)}|, 1\}\right)\right).$$

Then the modified height $h'(\alpha)$ is defined by

$$h'(\alpha) = \frac{1}{d}\max\{h(\alpha), |\log \alpha|, 1\},$$

where $h(\alpha) = d h_0(\alpha)$ is the standard logarithmic Weil height of α.

Let us consider the linear form

$$L(z_1, \ldots, z_k) = b_1 z_1 + \ldots + b_k z_k,$$

where $b_1, \ldots, b_k$ are rational integers, not all 0 and define

$$h'(L) = \frac{1}{d}\max\{h(L), 1\},$$

where $h(L) = d \log\left(\max_{1 \le j \le k}\left\{\frac{|b_j|}{b}\right\}\right)$ is the logarithmic Weil height of L, where b is the greatest common divisor of $b_1, \ldots, b_k$. If we write $B = \max\{|b_1|, \ldots, |b_k|, e\}$, then we get

$$h'(L) \leqslant \log B.$$

With these notations we are able to state the following result due to Baker and Wüstholz [3].

Theorem 2. *If* $\Lambda = L(\log \alpha_1, \ldots, \log \alpha_k) \neq 0$, *then*

$$\log |\Lambda| \geqslant -C(k,d) h'(\alpha_1) \ldots h'(\alpha_k) h'(L),$$

where

$$C(k,d) = 18(k+1)!\, k^{k+1} (32d)^{k+2} \log(2kd).$$

With $|\Lambda| \leqslant \frac{1}{2}$, we have $\frac{1}{2}|\Lambda| \leqslant |\Phi| \leqslant 2|\Lambda|$, where

$$\Phi = e^\Lambda - 1 = \alpha_1^{b_1} \cdots \alpha_k^{b_k} - 1,$$

so that

$$\log |\alpha_1^{b_1} \cdots \alpha_k^{b_k} - 1| \geqslant \log |\Lambda| - \log 2.$$

We apply Theorem 2 mainly in a situation where $k = 3$ and $d = 6$. In this case we obtain

$$C(3,6) = 18(4!)3^4(32 \times 6)^5(\log 36) \approx 3.2718\ldots \times 10^{16}.$$

We will use this value throughout the paper without any further reference.

2.2. **A generalized result by Dujella and Pethő.** The following result will be used to reduce the huge upper bounds for n and m which appear during the course of the proof of Theorem 1 (cf. Proposition 1). The following Lemma is stated in [7], which is regarded as a slight variation of a result due to Dujella and Pethő [8], of which is a generalization of a result due to Baker and Davenport [2]. For a real number x, let $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$ be the distance from x to the nearest integer.

Lemma 1. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M\|\tau q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < m\tau - n + \mu < AB^{-k},$$

in positive integers m, n and k with

$$m \leqslant M \qquad \text{and} \qquad k \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Proof of Theorem 1

3.1. **Set up.** Assume that $(n,m) \neq (n_1, m_1)$ are pairs of indices such that

$$F_n - F_{n_1} = T_m - T_{m_1}. \tag{3}$$

We may assume that $m \neq m_1$, since otherwise $(n,m) = (n_1, m_1)$. Furthermore we assume that $m > m_1$. Due to equation (3) and since the right hand side of equation (3) is positive, we get that the left hand side of equation (3) is also positive and thus $n > n_1$. Therefore, we have $n \geqslant 3$, $n_1 \geqslant 2$ and $m \geqslant 3$, $m_1 \geqslant 2$.

During the proof of Theorem 1 we use the Binet formulae for the Fibonacci sequence and Tribonacci sequence in the following form:

Fibonacci sequence:

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \qquad \text{for all } k \geqslant 0,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$. Besides, the inequality

$$\alpha^{k-2} \leqslant F_k \leqslant \alpha^{k-1}$$

holds for all $k \geqslant 1$.

Tribonacci sequence:

$$T_k = c_\alpha \alpha_T^k + c_\beta \beta_T^k + c_\gamma \gamma_T^k \qquad \text{for all } k \geqslant 0,$$

where α_T, β_T and γ_T are the roots of the characteristic equation $x^3 - x^2 - x - 1 = 0$, with

$$\alpha_T = \frac{1}{3}\left(1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}\right),$$

$$\beta_T = \frac{1}{6}\left(2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right) + \frac{\sqrt{3}}{6} i \left(\sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right),$$

$$\gamma_T = \frac{1}{6}\left(2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right) - \frac{\sqrt{3}}{6} i \left(\sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right),$$

and the coefficients

$$c_\alpha = \frac{\alpha_T}{(\alpha_T-\beta_T)(\alpha_T-\gamma_T)} = \frac{1}{-\alpha_T^2+4\alpha_T-1},$$
$$c_\beta = \frac{\beta_T}{(\beta_T-\alpha_T)(\beta_T-\gamma_T)} = \frac{1}{-\beta_T^2+4\beta_T-1},$$
$$c_\gamma = \frac{\gamma_T}{(\gamma_T-\alpha_T)(\gamma_T-\beta_T)} = \frac{1}{-\gamma_T^2+4\gamma_T-1}$$

are the roots of the polynomial $44x^3-2x-1$. Note that

$$\begin{array}{ll} 1.839 < \alpha_T < 1.840 & 0.336 < c_\alpha < 0.337 \\ \beta_T = \overline{\gamma_T} & 0.737 < |\beta_T| = |\gamma_T| < 0.738 \\ c_\beta = \overline{c_\gamma} & 0.259 < |c_\beta| = |c_\gamma| < 0.260. \end{array}$$

Finally let us state several useful inequalities. For instance

$$\alpha_T^{k-2} \leqslant T_k \leqslant \alpha_T^{k-1} \quad \text{for all } k \geqslant 1.$$

which was already shown in [6]. Using equation (3) we get that

$$\alpha^{n-4} \leqslant F_{n-2} \leqslant F_n - F_{n_1} = T_m - T_{m_1} < T_m \leqslant \alpha_T^{m-1}, \tag{4}$$

and similarly we get

$$\begin{aligned} \alpha^{n-1} \geqslant F_n > F_n - F_{n_1} = T_m - T_{m_1} \geqslant T_m - T_{m-1} = \\ T_{m-2} + T_{m-3} \geqslant \alpha_T^{m-4} + \alpha_T^{m-5} > 2.83\alpha_T^{m-5}. \end{aligned} \tag{5}$$

Thus

$$n-4 < \frac{\log \alpha_T}{\log \alpha}(m-1) \quad \text{and} \quad n-3 > \frac{\log \alpha_T}{\log \alpha}(m-5), \tag{6}$$

where $\dfrac{\log \alpha_T}{\log \alpha} \approx 1.2663\ldots$.

Inequality (6) implies that if $n < 300$, then $m < 240$. By a brute force computer enumeration for $2 \leqslant n_1 < n < 300$ and $2 \leqslant m_1 < m < 240$ we found all solutions listed in Theorem 1. Thus we may assume from now on that $n \geq 300$.

3.2. **Linear forms in logarithms.** Since $n \geq 300$, by the first inequality of (6) we obtain that $m \geq 235$ which combined with the second inequality of (6) implies that $n > m$. Moreover, we have

$$\frac{\alpha^n-\beta^n}{\sqrt{5}} - \frac{\alpha^{n_1}-\beta^{n_1}}{\sqrt{5}} = \left(c_\alpha\alpha_T^m + c_\beta\beta_T^m + c_\gamma\gamma_T^m\right) - \left(c_\alpha\alpha_T^{m_1} + c_\beta\beta_T^{m_1} + c_\gamma\gamma_T^{m_1}\right).$$

Collecting the "large" terms on the left hand side of the equation we obtain

$$\begin{aligned} \left|\frac{\alpha^n}{\sqrt{5}} - c_\alpha\alpha_T^m\right| &= \left|\frac{\beta^n}{\sqrt{5}} + \frac{\alpha^{n_1}-\beta^{n_1}}{\sqrt{5}} + (c_\beta\beta_T^m + c_\gamma\gamma_T^m) - (c_\alpha\alpha_T^{m_1} + c_\beta\beta_T^{m_1} + c_\gamma\gamma_T^{m_1})\right| \\ &\leqslant \frac{\alpha^{n_1}}{\sqrt{5}} + c_\alpha\alpha_T^{m_1} + \frac{|\beta|^n}{\sqrt{5}} + \frac{|\beta|^{n_1}}{\sqrt{5}} + |c_\beta|\,|\beta_T|^m + |c_\gamma|\,|\gamma_T|^m \\ &\quad + |c_\beta|\,|\beta_T|^{m_1} + |c_\gamma|\,|\gamma_T|^{m_1} \\ &< \frac{\alpha^{n_1}}{\sqrt{5}} + c_\alpha\alpha_T^{m_1} + 0.46 \\ &< 0.92\max\{\alpha^{n_1}, \alpha_T^{m_1}\}. \end{aligned}$$

Dividing by $c_\alpha \alpha_T^m$ we get

$$\left|(\sqrt{5}c_\alpha)^{-1}\alpha^n\alpha_T^{-m} - 1\right| < \max\left\{\frac{0.92}{c_\alpha\alpha_T^m}\alpha^{n_1}, \frac{0.92}{c_\alpha}\alpha_T^{m_1-m}\right\}$$
$$< \max\left\{2.74\frac{\alpha^{n_1}}{\alpha_T}\frac{1}{\alpha^{n-4}}, 2.74\alpha_T^{m_1-m}\right\}.$$

Hence we obtain the inequality

$$\left|(\sqrt{5}c_\alpha)^{-1}\alpha^n\alpha_T^{-m} - 1\right| < \max\{\alpha^{n_1-n+5}, \alpha_T^{m_1-m+2}\}. \tag{7}$$

Let us introduce

$$\Lambda = n\log\alpha - m\log\alpha_T - \log(\sqrt{5}c_\alpha)$$

and assume that $|\Lambda| \leqslant 0.5$. Further, we put

$$\Phi = e^\Lambda - 1 = (\sqrt{5}c_\alpha)^{-1}\alpha^n\alpha_T^{-m} - 1$$

and use the theorem of Baker and Wüstholz (Theorem 2) with the data

$$k = 3, \quad \alpha_1 = \sqrt{5}c_\alpha, \quad b_1 = -1, \quad \alpha_2 = \alpha, \quad b_2 = n, \quad \alpha_3 = \alpha_T, \quad b_3 = -m.$$

With this data we have $K = \mathbb{Q}(\sqrt{5}, \alpha_T)$, i.e. $d = 6$, and $B = n$. Notice that the minimal polynomial of α_1 is $1936x^6 - 880x^4 + 100x^2 - 125$, and we conclude that $h'(\alpha_1) = \frac{1}{6}\log 1936$. Further we obtain by a simple computation that $h'(\alpha_2) = \frac{1}{2}\log\alpha$ and $h'(\alpha_3) = \frac{1}{3}\log\alpha_T$.

Before we can apply Theorem 2 we have to show that $\Phi \neq 0$. Assume to the contrary that $\Phi = 0$, then $\alpha^{2n} = 5c_\alpha^2\alpha_T^{2m}$. But $\alpha^{2n} \in \mathbb{Q}(\sqrt{5})\setminus\mathbb{Q}$ whereas $5c_\alpha^2\alpha_T^{2m} \in \mathbb{Q}(\alpha_T)$. Thus $\Phi = 0$ is impossible due to the fact that $\mathbb{Q}(\sqrt{5}) \cap \mathbb{Q}(\alpha_T) = \mathbb{Q}$.

Apply Theorem 2 yields

$$\log|\Phi| \geqslant -C(3,6)\left(\frac{1}{6}\log 1936\right)\left(\frac{1}{2}\log\alpha\right)\left(\frac{1}{3}\log\alpha_T\right)\log n - \log 2$$

and together with inequality (7) we have

$$\min\{(n-n_1-5)\log\alpha, (m-m_1-2)\log\alpha_T\} \leqslant 2.02\times 10^{15}\log n.$$

Thus we have proved so far:

Lemma 2. *Assume that (n, m, n_1, m_1) is a solution to equation (3) with $m > m_1$. Then we have*

$$\min\{(n-n_1)\log\alpha, (m-m_1)\log\alpha_T\} < 2.03\times 10^{15}\log n.$$

Note that in the case that $|\Lambda| > 0.5$ inequality (7) is possible only if either $n - n_1 \leqslant 5$ or $m - m_1 \leqslant 2$, which is covered by the bound provided by Lemma 2.

Now we have to distinguish between the following two cases:

Case 1. Let us assume that

$$\min\{(n-n_1)\log\alpha, (m-m_1)\log\alpha_T\} = (n-n_1)\log\alpha.$$

We rewrite equation (3) as

$$\begin{aligned}\left|\frac{\alpha^n-\alpha^{n_1}}{\sqrt{5}}-c_\alpha\alpha_T^m\right| &= \Bigg|-c_\alpha\alpha_T^{m_1}+\frac{\beta^n}{\sqrt{5}}-\frac{\beta^{n_1}}{\sqrt{5}}+\\ &\qquad (c_\beta\beta_T^m+c_\gamma\gamma_T^m)-(c_\beta\beta_T^{m_1}+c_\gamma\gamma_T^{m_1})\Bigg|\\ &\leqslant c_\alpha\alpha_T^{m_1}+\frac{|\beta^n|}{\sqrt{5}}+\frac{|\beta^{n_1}|}{\sqrt{5}}+|c_\beta|\,|\beta_T^m|+|c_\gamma|\,|\gamma_T^m|+\\ &\quad |c_\beta|\,|\beta_T^{m_1}|+|c_\gamma|\,|\gamma_T^{m_1}|\end{aligned}$$

and obtain that

$$\left|\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\alpha^{n_1}-c_\alpha\alpha_T^m\right|<(c_\alpha+0.14)\alpha_T^{m_1}.$$

Dividing by $c_\alpha\alpha_T^m$ we get the inequality

$$\left|\frac{\alpha^{n-n_1}-1}{\sqrt{5}c_\alpha}\alpha^{n_1}\alpha_T^{-m}-1\right|<1.42\alpha_T^{m_1-m}. \tag{8}$$

Case 2. Let us assume that

$$\min\{(n-n_1)\log\alpha,(m-m_1)\log\alpha_T\}=(m-m_1)\log\alpha_T.$$

We rewrite equation (3) as

$$\begin{aligned}\left|\frac{\alpha^n}{\sqrt{5}}-c_\alpha\alpha_T^m+c_\alpha\alpha_T^{m_1}\right| &= \left|\frac{\beta^n}{\sqrt{5}}+\frac{\alpha^{n_1}-\beta^{n_1}}{\sqrt{5}}+c_\beta\beta_T^m+c_\gamma\gamma_T^m-c_\beta\beta_T^{m_1}-c_\gamma\gamma_T^{m_1}\right|\\ &\leqslant \frac{|\beta^n|}{\sqrt{5}}+\frac{\alpha^{n_1}}{\sqrt{5}}+\frac{|\beta^{n_1}|}{\sqrt{5}}+|c_\beta|\,|\beta_T^m|+|c_\gamma|\,|\gamma_T^m|\\ &\quad +|c_\beta|\,|\beta_T^{m_1}|+|c_\gamma|\,|\gamma_T^{m_1}|.\end{aligned}$$

Thus we get

$$\left|\alpha^n-\sqrt{5}c_\alpha(\alpha_T^{m-m_1}-1)\alpha_T^{m_1}\right|<1.4\alpha^{n_1}.$$

Dividing both sides by $\sqrt{5}c_\alpha(\alpha_T^{m-m_1}-1)\alpha_T^{m_1}$ we get by using inequality (4) the following inequality:

$$\left|\frac{\alpha^n\alpha_T^{-m_1}}{\sqrt{5}\,c_\alpha(\alpha_T^{m-m_1}-1)}-1\right|<\frac{1.4}{\sqrt{5}\,c_\alpha(1-\alpha_T^{m_1-m})\alpha_T}\frac{\alpha^{n_1}}{\alpha_T^{m-1}}<2.22\alpha^{n_1-n+4}. \tag{9}$$

We want to apply Theorem 2 to both inequalities (8) and (9) respectively. Let us consider the first case more closely. We write

$$\Lambda_1=n_1\log\alpha-m\log\alpha_T+\log\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}c_\alpha}\right)$$

and assume that $|\Lambda_1|\leqslant 0.5$. Further, we put

$$\Phi_1=e^{\Lambda_1}-1=\frac{\alpha^{n-n_1}-1}{\sqrt{5}c_\alpha}\alpha^{n_1}\alpha_T^{-m}-1$$

and aim to apply Theorem 2 by taking $K=\mathbb{Q}(\sqrt{5},\alpha_T)$, i.e. $d=6$, $k=3$ and $B=n$. Further, we have

$$\alpha_1=\frac{\alpha^{n-n_1}-1}{\sqrt{5}c_\alpha},\quad b_1=1,\quad \alpha_2=\alpha,\quad b_2=n_1,\quad \alpha_3=\alpha_T,\quad b_3=-m.$$

Let us estimate the height of α_1. Notice that $h(\alpha_1) \leqslant h(\eta_1) + h(\eta_2)$, where $\eta_1 = \frac{\alpha^{n-n_1}-1}{\sqrt{5}}$ and $\eta_2 = \frac{1}{c_\alpha}$. The minimal polynomial of η_1 divides (e.g. see [7])

$$5X^2 - 5F_{n-n_1}X - ((-1)^{n-n_1} + 1 - L_{n-n_1}),$$

where $\{L_k\}_{k\geqslant 0}$ is the Lucas companion sequence of the Fibonacci sequence given by $L_0 = 2, L_1 = 1, L_{k+2} = L_{k+1} + L_k$ for $k \geqslant 0$. Its Binet formula for the general term is $L_k = \alpha^k + \beta^k$ for all $k \geqslant 0$. Thus (cf. [7]),

$$h_0(\eta_1) \leqslant \frac{1}{2}\left(\log 5 + \log\left(\frac{\alpha^{n-n_1}+1}{\sqrt{5}}\right)\right).$$

Thus Lemma 2 yields an upper bound

$$h_0(\eta_1) < \frac{1}{2}\log\left(2\sqrt{5}\alpha^{n-n_1}\right) < \frac{1}{2}(n - n_1 + 4)\log\alpha < 1.02 \times 10^{15}\log n,$$

i.e. $h(\eta_1) < 6 \times 1.02 \times 10^{15}\log n$. Since $h_0(\eta_2) = h_0(c_\alpha) = \frac{1}{3}\log 44$, i.e. $h(\eta_2) = 2\log 44$, we have $h(\alpha_1) \leqslant 6 \times 1.02 \times 10^{15}\log n + 2\log 44$ and finally we obtain that

$$h'(\alpha_1) < 1.03 \times 10^{15}\log n.$$

Moreover, we have that $h'(\alpha_2) = \frac{1}{2}\log\alpha$ and $h'(\alpha_3) = \frac{1}{3}\log\alpha_T$ as before.

Now let us turn to the second case. We write

$$\Lambda_2 = n\log\alpha - m_1\log\alpha_T - \log\left(\sqrt{5}\,c_\alpha(\alpha_T^{m-m_1} - 1)\right)$$

and assume that $|\Lambda_2| \leqslant 0.5$. Further, we put

$$\Phi_2 = e^{\Lambda_2} - 1 = (\sqrt{5}\,c_\alpha(\alpha_T^{m-m_1} - 1))^{-1}\alpha^n\alpha_T^{-m_1} - 1$$

and aim to apply Theorem 2. As in the previous case we take $K = \mathbb{Q}(\sqrt{5}, \alpha_T)$, i.e. $d = 6$, $k = 3$ and $B = n$. Further, we have

$$\alpha_1 = \sqrt{5}\,c_\alpha(\alpha_T^{m-m_1} - 1), \quad b_1 = -1, \quad \alpha_2 = \alpha, \quad b_2 = n, \quad \alpha_3 = \alpha_T, \quad b_3 = -m_1.$$

Again, we have to estimate $h(\alpha_1)$ and therefore note that $h(\alpha_1) \leqslant h(\eta_1) + h(\eta_2) + h(\eta_3)$, where $\eta_1 = \alpha_T^{m-m_1} - 1$, $\eta_2 = c_\alpha$ and $\eta_3 = \sqrt{5}$. By applying Lemma 2 we get

$$\begin{aligned} h_0(\eta_1) &\leqslant h_0(\alpha_T^{m-m_1}) + h_0(-1) + \log 2 \\ &= (m - m_1)h_0(\alpha_T) + \log 2 = \frac{m - m_1}{3}\log\alpha_T + \log 2 \\ &< \frac{1}{3} \times 2.03 \times 10^{15}\log n + \log 2. \end{aligned}$$

Thus

$$h(\alpha_1) < 6\left(\frac{1}{3} \times 2.03 \times 10^{15}\log n + \log 2 + \frac{1}{3}\log 44 + \log\sqrt{5}\right)$$

and therefore

$$h'(\alpha_1) < 6.77 \times 10^{14}\log n < 1.03 \times 10^{15}\log n.$$

Once again, we have that $h'(\alpha_2) = \frac{1}{2}\log\alpha$ and $h'(\alpha_3) = \frac{1}{3}\log\alpha_T$.

In particular, we have shown in both cases that

$$h'(\alpha_1) < 1.03 \times 10^{15}\log n, \quad h'(\alpha_2) = \frac{1}{2}\log\alpha, \quad h'(\alpha_3) = \frac{1}{3}\log\alpha_T, \quad B = n.$$

But, before we can apply Theorem 2 we have to ensure that $\Phi_i \neq 0$ for $i = 1, 2$. Firstly we deal with the assumption that $\Phi_1 = 0$, i.e. $\alpha^n - \alpha^{n_1} = \sqrt{5}c_\alpha\alpha_T^m$. This is impossible if $\sqrt{5}c_\alpha\alpha_T^m \in \mathbb{Q}(\sqrt{5}, \alpha_T)$ but $\notin \mathbb{Q}(\sqrt{5})$. Therefore let us assume that

$\sqrt{5}c_\alpha\alpha_T^m \in \mathbb{Q}(\sqrt{5})$, hence $\sqrt{5}c_\alpha\alpha_T^m = y\sqrt{5}$ for some $y \in \mathbb{Q}$. Let $\sigma \neq \mathrm{id}$ be the unique non-trivial $\mathbb{Q}$-automorphism over $\mathbb{Q}(\sqrt{5})$. Then we get

$$\alpha^n - \alpha^{n_1} = \sqrt{5}c_\alpha\alpha_T^m = y\sqrt{5} = -\sigma(\sqrt{5}c_\alpha\alpha_T^m) = -\sigma(\alpha^n - \alpha^{n_1}) = \beta^{n_1} - \beta^n.$$

However, the absolute value of $\alpha^n - \alpha^{n_1}$ is at least $\alpha^n - \alpha^{n_1} \geqslant \alpha^{n-2} \geqslant \alpha^{298} > 2$ whereas the absolute value of $\beta^{n_1} - \beta^n$ is at most $|\beta^{n_1} - \beta^n| \leqslant |\beta|^{n_1} + |\beta|^n < 2$. By this obvious contradiction we conclude that $\Phi_1 \neq 0$.

Now let us consider the second case and assume for the moment that $\Phi_2 = 0$, i.e. $\alpha^{2n} = 5\alpha_T^{2m_1}c_\alpha^2(\alpha_T^{m-m_1} - 1)^2$. However, $\alpha^{2n} \in \mathbb{Q}(\sqrt{5}) \setminus \mathbb{Q}$, whereas $5\alpha_T^{2m_1}c_\alpha^2(\alpha_T^{m-m_1} - 1)^2 \in \mathbb{Q}(\alpha_T)$. Thus we obtain also in this case a contradiction and we also conclude in this case that $\Phi_2 \neq 0$.

Now, we are ready to apply Theorem 2 and get

$$\begin{aligned}\log|\Phi_i| &> -C(3,6)\left(1.03 \times 10^{15}\log n\right)\left(\frac{1}{2}\log\alpha\right)\left(\frac{1}{3}\log\alpha_T\right)\log n - \log 2\\ &> -1.65 \times 10^{30}(\log n)^2\end{aligned}$$

for $i = 1, 2$. Combining this inequality with the inequalities (8) and (9), we obtain

$$(m_1 - m)\log\alpha_T + \log 1.42 > -1.65 \times 10^{30}(\log n)^2$$

and

$$(n_1 - n + 4)\log\alpha + \log 2.22 > -1.65 \times 10^{30}(\log n)^2$$

respectively. These two inequalities yield together with Lemma 2 the following lemma:

Lemma 3. *Assume that (n, m, n_1, m_1) is a solution to equation (3) with $m > m_1$. Then we have*

$$\max\{(n - n_1)\log\alpha, (m - m_1)\log\alpha_T\} < 1.66 \times 10^{30}(\log n)^2.$$

Note that in view of inequality (8) $|\Lambda_1| > 0.5$ is possible only if $m - m_1 = 1$ and in view of inequality (9) $|\Lambda_2| > 0.5$ is possible only if $n - n_1 \leqslant 6$ respectively. Both cases are covered by the bound provided by Lemma 3.

One more time we have to apply Theorem 2. This time we rewrite equation (3) as

$$\begin{aligned}&\left|\frac{\alpha^n}{\sqrt{5}} - \frac{\alpha^{n_1}}{\sqrt{5}} - c_\alpha\alpha_T^m + c_\alpha\alpha_T^{m_1}\right|\\ &\qquad = \left|\frac{\beta^n}{\sqrt{5}} - \frac{\beta^{n_1}}{\sqrt{5}} + c_\beta\beta_T^m + c_\gamma\gamma_T^m - c_\beta\beta_T^{m_1} - c_\gamma\gamma_T^{m_1}\right| < 0.46\end{aligned}$$

Dividing both sides by $c_\alpha\alpha_T^{m_1}(\alpha_T^{m-m_1} - 1)$ we get by applying inequality (4)

$$\left|\frac{\alpha^{n-n_1} - 1}{\sqrt{5}\,c_\alpha(\alpha_T^{m-m_1} - 1)}\alpha^{n_1}\alpha_T^{-m_1} - 1\right| < \frac{0.46}{c_\alpha(1 - \alpha_T^{m_1-m})\alpha_T}\frac{1}{\alpha_T^{m-1}} < 1.64\alpha^{4-n}. \tag{10}$$

In this final step we consider the linear form

$$\Lambda_3 = n_1\log\alpha - m_1\log\alpha_T + \log\left(\frac{\alpha_T^{n-n_1} - 1}{\sqrt{5}\,c_\alpha(\alpha_T^{m-m_1} - 1)}\right)$$

and assume that $|\Lambda_3| \leqslant 0.5$. Further, we put

$$\Phi_3 = e^{\Lambda_3} - 1 = \frac{\alpha^{n-n_1} - 1}{\sqrt{5}\, c_\alpha (\alpha_T^{m-m_1} - 1)} \alpha^{n_1} \alpha_T^{-m_1} - 1.$$

As before we take $K = \mathbb{Q}(\sqrt{5}, \alpha_T)$, i.e. $d = 6$, $k = 3$, $B = n$ and we have

$$\alpha_1 = \frac{\alpha^{n-n_1} - 1}{\sqrt{5}\, c_\alpha (\alpha_T^{m-m_1} - 1)}, \quad b_1 = 1, \quad \alpha_2 = \alpha, \quad b_2 = n_1, \quad \alpha_3 = \alpha_T, \quad b_3 = -m_1.$$

By Lemma 3 and similar computations as done before we obtain that

$$h\left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5} c_\alpha}\right) \leqslant 6\left(\frac{1}{2}(n - n_1 + 4)\log\alpha\right) + 2\log 44 < 3 \times \left(1.67 \times 10^{30} (\log n)^2\right)$$

and

$$h(\alpha_T^{m-m_1} - 1) \leqslant 6\left(\frac{m - m_1}{3} \log\alpha_T + \log 2\right) < 2 \times \left(1.67 \times 10^{30} (\log n)^2\right).$$

Therefore we find the upper bound

$$h(\alpha_1) \leqslant h(\frac{\alpha^{n-n_1} - 1}{\sqrt{5} c_\alpha}) + h(\alpha_T^{m-m_1} - 1) < 5 \times \left(1.67 \times 10^{30} (\log n)^2\right)$$

and thus

$$h'(\alpha_1) < \frac{5}{6} \times \left(1.67 \times 10^{30} (\log n)^2\right).$$

As before, we have $h'(\alpha_2) = \dfrac{1}{2}\log\alpha$ and $h'(\alpha_3) = \dfrac{1}{3}\log\alpha_T$.

Using similar arguments as in the proof that $\Phi_1 \neq 0$ we can show that $\Phi_3 \neq 0$. Now an application of Theorem 2 yields

$$\log|\Phi_3| > -C(3, 6)\left(\frac{5}{6} \times 1.67 \times 10^{30} (\log n)^2\right)\left(\frac{1}{2}\log\alpha\right)\left(\frac{1}{3}\log\alpha_T\right)\log n - \log 2.$$

Combining this inequality with inequality (10) we get

$$(n - 4)\log\alpha < 2.23 \times 10^{45} (\log n)^3$$

which yields $n < 8 \times 10^{51}$.

Similarly as in the cases above the assumption that $|\Lambda_3| > 0.5$ leads in view of inequality (10) to $n \leqslant 5$. Let us summarize the results of this subsection:

Proposition 1. *Assume that* (n, m, n_1, m_1) *is a solution to equation* (3) *with* $m > m_1$. *Then we have that* $n < 8 \times 10^{51}$.

Remark 1. The theorem of Baker and Wüstholz (Theorem 2) can be easily applied. However, a slightly sharper bound for n, namely $n < 6 \times 10^{48}$, may be obtained if one uses Matveev's result [9] instead. However, the improvement is not crucial in view of our next step, the reduction of our upper bound for n using the method of Baker and Davenport (Lemma 1).

3.3. **Generalized method of Baker and Davenport.** In our final step we reduce the huge upper bound for n form Proposition 1 by applying several times Lemma 1. In this subsection we follow the ideas from [7]. First, we consider inequality (7) and recall that

$$\Lambda = n\log\alpha - m\log\alpha_T - \log(\sqrt{5}c_\alpha).$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geqslant 20$. In the case that this condition fails we consider one of the following inequalities instead:

- if $n-n_1 < 20$ but $m-m_1 \geqslant 20$, we consider inequality (8);
- if $n-n_1 \geqslant 20$ but $m-m_1 < 20$, we consider inequality (9);
- if both $n-n_1 < 20$ and $m-m_1 < 20$, we consider inequality (10).

Let us start by considering inequality (7). Since we assume that $\min\{n-n_1, m-m_1\} \geqslant 20$ we get $|\Phi| = |e^\Lambda - 1| < \frac{1}{4}$, hence $|\Lambda| < \frac{1}{2}$. And, since $|x| < 2|e^x-1|$ holds for all $x \in (-\frac{1}{2}, \frac{1}{2})$ we get

$$|\Lambda| < 2\max\{\alpha^{n_1-n+5}, \alpha_T^{m_1-m+2}\} \leqslant \max\{\alpha^{n_1-n+7}, \alpha_T^{m_1-m+4}\}.$$

Assume that $\Lambda > 0$. Then we have the inequality

$$\begin{aligned} 0 < n\left(\frac{\log\alpha}{\log\alpha_T}\right) - m + \frac{\log(1/(\sqrt{5}c_\alpha))}{\log\alpha_T} &< \max\left\{\frac{\alpha^7}{\log\alpha_T}\alpha^{-(n-n_1)}, \frac{\alpha_T^4}{\log\alpha_T}\alpha_T^{-(m-m_1)}\right\} \\ &< \max\left\{48\alpha^{-(n-n_1)}, 19\alpha_T^{-(m-m_1)}\right\} \end{aligned}$$

and we apply Lemma 1 with

$$\tau = \frac{\log\alpha}{\log\alpha_T}, \quad \mu = \frac{\log(1/(\sqrt{5}c_\alpha))}{\log\alpha_T}, \quad (A,B) = (48,\alpha) \text{ or } (19,\alpha_T).$$

Let $\tau = [a_0, a_1, a_2, \dots] = [0,1,3,1,3,13,2,1,8,3,1,5,\dots]$ be the continued fraction of τ. Moreover, we choose $M = 8 \times 10^{51}$ and consider the 104-th convergent

$$\frac{p}{q} = \frac{p_{104}}{q_{104}} = \frac{52841963647885529119220800813840965784230907639792403 3}{66915901128412992013946827929750445311260816077167181 0},$$

with $q = q_{104} > 6M$. This yields $\varepsilon > 0.068$ and therefore either

$$n - n_1 \leqslant \frac{\log(48q/0.068)}{\log\alpha} < 272, \quad \text{or} \quad m - m_1 \leqslant \frac{\log(19q/0.068)}{\log\alpha_T} < 213.$$

Thus, we have either $n - n_1 \leqslant 271$, or $m - m_1 \leqslant 212$.

In the case of $\Lambda < 0$ we consider the following inequality:

$$\begin{aligned} 0 < m\left(\frac{\log\alpha_T}{\log\alpha}\right) - n + \frac{\log(\sqrt{5}c_\alpha)}{\log\alpha} &< \max\left\{\frac{\alpha^7}{\log\alpha_T}\alpha^{-(n-n_1)}, \frac{\alpha_T^4}{\log\alpha_T}\alpha_T^{-(m-m_1)}\right\} \\ &< \max\left\{61\alpha^{-(n-n_1)}, 24\alpha_T^{-(m-m_1)}\right\} \end{aligned}$$

instead and apply Lemma 1 with

$$\tau = \frac{\log\alpha_T}{\log\alpha}, \quad \mu = \frac{\log(\sqrt{5}c_\alpha)}{\log\alpha}, \quad (A,B) = (61,\alpha) \text{ or } (24,\alpha_T).$$

Let $\tau = [a_0, a_1, a_2, \dots] = [1,3,1,3,13,2,1,8,3,1,5,2,\dots]$ be the continued fraction of τ. Again, we choose $M = 8 \times 10^{51}$ but in this case we consider instead of the 104-th convergent the 103-rd convergent

$$\frac{p}{q} = \frac{p_{103}}{q_{103}} = \frac{669159011284129920139468279297504453112608160771671810}{528419636478855291192208008138409657842309076397924033},$$

with $q > 6M$. This yields $\varepsilon > 0.067$ and again we obtain either

$$n - n_1 \leqslant \frac{\log(61q/0.067)}{\log \alpha} < 272, \quad \text{or} \quad m - m_1 \leqslant \frac{\log(24q/0.067)}{\log \alpha_T} < 213.$$

These bounds agree with the bounds obtained in the case that $\Lambda > 0$. As a conclusion, we have either $n - n_1 \leqslant 271$ or $m - m_1 \leqslant 212$ whenever $\Lambda \neq 0$.

Now, we have to distinguish between the two cases $n - n_1 \leqslant 271$ and $m - m_1 \leqslant 212$. First, let us assume that $n - n_1 \leqslant 271$. In this case we consider inequality (8) and assume that $m - m_1 \geqslant 20$. Recall that

$$\Lambda_1 = n_1 \log \alpha - m \log \alpha_T + \log\left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}c_\alpha}\right)$$

and inequality (8) yields that

$$|\Lambda_1| < \alpha_T^{m_1 - m + 2}.$$

If we further assume that $\Lambda_1 > 0$, then we get

$$0 < n_1\left(\frac{\log \alpha}{\log \alpha_T}\right) - m + \frac{\log((\alpha^{n-n_1} - 1)/(\sqrt{5}c_\alpha))}{\log \alpha_T} < \frac{\alpha_T^2}{\log \alpha_T}\alpha_T^{-(m-m_1)} < 6\alpha_T^{-(m-m_1)}.$$

Again we apply Lemma 1 with the same τ and M as in the case that $\Lambda > 0$. We use the 104-th convergent $\frac{p}{q} = \frac{p_{104}}{q_{104}}$ of τ as before. But, in this case we choose $(A, B) = (6, \alpha_T)$ and use

$$\mu_k = \frac{\log((\alpha^k - 1)/(\sqrt{5}c_\alpha))}{\log \alpha_T},$$

instead of μ for each possible value of $n - n_1 = k = 1, 2, \ldots, 271$. A quick computer aid computation yields that $\varepsilon > 0.00038$ for all $1 \leqslant k \leqslant 271$. Hence, by Lemma 1, we get

$$m - m_1 < \frac{\log(6q/0.00038)}{\log \alpha_T} < 220.$$

Thus, $n - n_1 \leqslant 271$ implies $m - m_1 \leqslant 219$.

In the case that $\Lambda_1 < 0$ we follow the ideas from the case that $\Lambda_1 > 0$. We use the same τ as in the case that $\Lambda < 0$ but instead of μ we take

$$\mu_k = \frac{\log(\sqrt{5}c_\alpha/(\alpha^k - 1))}{\log \alpha}$$

for each possible value of $n - n_1 = k = 1, 2, \ldots, 271$. Using Lemma 1 with this setting we also obtain in this case that $n - n_1 \leqslant 271$ implies $m - m_1 \leqslant 219$.

Now let us turn to the case that $m - m_1 \leqslant 212$ and let us consider inequality (9). Recall that

$$\Lambda_2 = n \log \alpha - m_1 \log \alpha_T + \log\left(\frac{1}{\sqrt{5}\,c_\alpha(\alpha_T^{m-m_1} - 1)}\right)$$

and let us assume that $n - n_1 \geqslant 20$. Then we have

$$|\Lambda_2| < \frac{4.44\alpha^4}{\alpha^{n-n_1}}.$$

Assuming that $\Lambda_2 > 0$, we get

$$0 < n\left(\frac{\log \alpha}{\log \alpha_T}\right) - m_1 + \frac{\log(1/(\sqrt{5}\,c_\alpha(\alpha_T^{m-m_1} - 1)))}{\log \alpha_T} < \frac{4.44\alpha^4}{(\log \alpha_T)\alpha^{n-n_1}} < 50\alpha^{-(n-n_1)}.$$

Once again we apply Lemma 1 with the same τ and M as for the case $\Lambda > 0$ before. We take $(A, B) = (50, \alpha)$ and

$$\mu_k = \frac{\log(1/(\sqrt{5}\, c_\alpha(\alpha_T^k - 1)))}{\log \alpha_T}$$

for every possible value $m - m_1 = k = 1, 2, \ldots, 212$. If we use again the 104-th convergent of τ, i.e. we put $q = q_{104}$, then for each k that yields a positive ε, we get $\varepsilon > 0.0012$. Therefore we get

$$n - n_1 < \frac{\log(50 q_{104}/0.0012)}{\log \alpha} < 280$$

in these cases. But for $k = 90$ we get a negative ε. In this case we consider the 105-th convergent $\frac{p}{q} = \frac{p_{105}}{q_{105}}$ of τ instead. Let us note that

$$q_{105} = 2012001397989667511935741474359297762971541412111 9669783.$$

Now we obtain in the case $k = 90$ that $\varepsilon > 0.46$. Thus

$$n - n_1 < \frac{\log(50 q_{105}/0.46)}{\log \alpha} < 275.$$

In the case that $\Lambda_2 < 0$ we follow again the ideas from the case that $\Lambda_2 > 0$. Of course we choose

$$\tau = \frac{\log \alpha_T}{\log \alpha} \quad \text{and} \quad \mu_k = \frac{\log(\sqrt{5}\, c_\alpha(\alpha_T^k - 1))}{\log \alpha}.$$

Applying Lemma 1 for all possible values of $m - m_1 = k = 1, \ldots, 212$ also yields in this case that $n - n_1 \leqslant 279$.

Let us summarize the above computations. First we got that either $n - n_1 \leqslant 271$, or $m - m_1 \leqslant 212$. If we assume that $n - n_1 \leqslant 271$, then we deduce that $m - m_1 \leqslant 219$, and if we assume that $m - m_1 \leqslant 212$, then we deduce that $n - n_1 \leqslant 279$. Altogether we obtain $n - n_1 \leq 279$ and $m - m_1 \leq 219$.

For the last step in our reduction process we consider inequality (10). Recall that

$$\Lambda_3 = n_1 \log \alpha - m_1 \log \alpha_T + \log\left(\frac{\alpha^{n-n_1} - 1}{\sqrt{5}\, c_\alpha(\alpha_T^{m-m_1} - 1)}\right).$$

Since we assume that $n \geqslant 300$, inequality (10) implies that

$$|\Lambda_3| < \frac{3.28\alpha^4}{\alpha^n}.$$

Let us assume that $\Lambda_3 > 0$. Then

$$0 < n_1 \left(\frac{\log \alpha}{\log \alpha_T}\right) - m_1 + \frac{\log\left((\alpha^k - 1)/(\sqrt{5}\, c_\alpha(\alpha_T^l - 1))\right)}{\log \alpha_T} < \frac{3.28\alpha^4}{(\log \alpha_T)\alpha^n} < 37\alpha^{-n},$$

where $(k, l) = (n - n_1, m - m_1)$. We apply Lemma 1 once more with the same τ and M as for the case when $\Lambda > 0$. Moreover, we take $(A, B) = (37, \alpha)$, and put

$$\mu_{k,l} = \frac{\log\left((\alpha^k - 1)/(\sqrt{5}\, c_\alpha(\alpha_T^l - 1))\right)}{\log \alpha_T}$$

for $1 \leqslant k \leqslant 279$ and $1 \leqslant l \leqslant 219$. We consider the 104-th convergent $\frac{p}{q} = \frac{p_{104}}{q_{104}}$. For all pairs (k,l) such that ε is positive we have indeed $\varepsilon > 2.8 \times 10^{-6}$. Thus for these pairs (k,l) Lemma 1 yields that

$$n \leqslant \frac{\log(37q_{104}/0.0000028)}{\log \alpha} < 292.$$

For all the remaining pairs (k,l) which yield a negative ε, we consider the 105-th convergent $\frac{p}{q} = \frac{p_{105}}{q_{105}}$ instead. And for all those pairs (k,l) the quantity ε is positive for this choice of q. In particular, we have that $\varepsilon > 0.0018$ for all these cases, hence

$$n \leqslant \frac{\log(37q_{105}/0.0018)}{\log \alpha} < 286.$$

In the case that $\Lambda_3 < 0$ the method is similar. In particular we have to apply Lemma 1 with

$$\tau = \frac{\log \alpha_T}{\log \alpha} \quad \text{and} \quad \mu_{k,l} = \frac{\log\left((\sqrt{5}\, c_\alpha(\alpha_T^l - 1))/(\alpha^k - 1)\right)}{\log \alpha}.$$

However, we obtain in this case the slightly smaller bound $n < 289$.

Altogether our reduction procedure yields the upper bound $n \leqslant 291$. However, this contradicts our assumption that $n \geqslant 300$. Thus Theorem 1 is proved.

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